Scalar solitons in a 4-Dimensional curved space-time

José A. González[1] and Daniel Sudarsky[2]
Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
A.P. 70-543, México, D.F. 04510, México.

Abstract

There is a theorem known as a Virial theorem that restricts the possible existence of non-trivial static solitary waves with scalar fields in a flat space-time with 3 or more spatial dimensions. This raises the following question: Does the analogous curved space-time version hold?. We investigate the possibility of solitons in a 4-D curved space-time with a simple model using numerical analysis. We found that there exists a static solution of the proposed non linear wave equation. This proves that in curved space-time the possibilities of solitonic solutions is enhanced relative to the flat space-time case.

PACS: 02.60.Cb; 02.60.Lj; 11.10.-z

1 Introduction

Solitons are special solutions of non-linear wave equations. The most relevant characteristic of solitons is that they are localized static solutions. The simplest example consist of a single scalar field ϕ in one spatial and one temporal dimensions. Perhaps the most famous one is the sine-Gordon soliton [1],[2].

At first sight, it can be thought that a wave equation with a single scalar field in more than three spatial dimensions with solitonic solutions can be found. However there is a Virial theorem which restricts this possibility. Here we are going to transcript that theorem and it's proof for the convenience of the reader [3]:

Theorem 1 *There are no non-trivial static solitary waves of systems with scalar fields when the space dimensionality is three or more and*

[1]cervera@nuclecu.unam.mx
[2]sudarsky@nuclecu.unam.mx

when the Lagrangian has the form:

$$\mathcal{L}(\mathbf{x}, \mathrm{t}) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \mathrm{U}(\phi(\mathbf{x}, \mathrm{t})) \tag{1}$$

with $\phi = [\phi_i(\mathbf{x}, \mathrm{t}); \mathrm{i} = 1, ..., \mathrm{N}]$ *a set of N coupled scalar fields in D space plus one time dimensions, and* $U(\phi(\mathbf{x}, \mathrm{t}))$ *a positive definite potential.*

Proof 1 *A static solution* $\phi(\mathbf{x})$ *obeys*

$$\nabla^2 \phi = \frac{\partial U}{\partial \phi}(\mathbf{x}) \tag{2}$$

where ∇^2 *is the Laplacian in D dimensions. This equation clearly correspond to the extremum condition* $\delta W = 0$ *for the static energy functional*

$$W[\phi] \equiv \int d^D x \left[\frac{1}{2} \nabla_i \phi \cdot \nabla_i \phi + U(\phi(\mathbf{x})) \right] \equiv V_1[\phi] + V_2[\phi] \tag{3}$$

where the functionals V_1 *and* V_2 *stand for the two terms on the right-hand side. Note that not only* W *but also* V_1 *and* V_2 *are non-negative. Now, let* $\phi_1(\mathbf{x})$ *be a static solution. Consider the one-parameter family of configurations*

$$\phi_\lambda = \phi_1(\lambda \mathbf{x}). \tag{4}$$

It is easy to check that

$$W[\phi_\lambda] = V_1[\phi_\lambda] + V_2[\phi_\lambda] = \lambda^{2-D} V_1[\phi_1] + \lambda^{-D} V_2[\phi_1]. \tag{5}$$

Since $\phi_1(\mathbf{x})$ *is an extremum of* $W[\phi]$*, it must in particular make* $W[\phi_\lambda]$ *satationary with respect to variations in* λ*; that is,*

$$\frac{d}{d\lambda} W[\phi_\lambda] = 0 \qquad at \quad \lambda = 1. \tag{6}$$

Differentiating (5) using (6) gives us

$$(2 - D) V_1[\phi_1] = D V_2[\phi_1]. \tag{7}$$

Since V_1 *and* V_2 *are non-negative (7) cannot be satisfied for* $D \geq 3$ *unless* $V_1[\phi_1] = V_2[\phi_1] = 0$*. This means that* $\phi_1(\mathbf{x})$ *has to be space-independent and equal to one of the zeros of* $U[\phi]$*. This is just a trivial solution and the theorem precludes non-trivial space-dependent solutions.* **q.e.d.**

For the case of $D = 2$, equation (7) tell us that $V_2[\phi_1] = 0$. The simplest example of a solution of this kind is the non-linear O(3) model [4],[5],[6] relevant in the description of the statical mechanics of an isotropic ferromagnet.

We are interested in the possible existence of a static solution of a system in a curved space-time in three spatial and one temporal dimensions. We will construct a simple model and look for solitonic solutions numerically.

2 The model

Consider the simplest case in a curved space-time. One scalar field whose equation of motion is:

$$\partial^\mu \partial_\mu \phi - \frac{\partial V}{\partial \phi} = 0 \tag{8}$$

Without potencial (8) is the Laplace equation,whose solutions take a maximum or minimum value only at the spatial boundaries. If we solve the Laplace equation in a space with a connected boundary or in a compact space, the solution in every point of the space will necessarily have the same value as in the boundary (i.e. a trivial solution).

So we are going to introduce a potential $V = (\phi^2 - 1)^2$ in analogy with the $1 + 1$ dimensional case. Moreover, we consider the simplest kind of spatialy compact universe, the static Einstein Universe:

$$ds^2 = -dt^2 + d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\varphi^2). \tag{9}$$

Equation (8) in the space-time corresponding to (9) takes the form:

$$\frac{\partial^2\phi}{\partial\chi^2} - \frac{\partial^2\phi}{\partial t^2} + 2cot\chi\frac{\partial\phi}{\partial\chi} + sin^2\chi cot\theta\frac{\partial\phi}{\partial\theta} + sin^2\chi\frac{\partial^2\phi}{\partial\theta^2} + sin^2\chi sin^2\theta\frac{\partial^2\phi}{\partial\varphi^2} = 2(\phi^2 - 1)2\phi. \tag{10}$$

We are interested in static solutions and with spherical simetry so we have

$$\frac{d^2\phi}{d\chi^2} + 2cot\chi\frac{d\phi}{d\chi} - 4\phi(\phi^2 - 1) = 0 \tag{11}$$

this is an ordinary differential equation of second order. Equation (11) can be separated in 2 ordinary equations of first order. Let:

$$x_1 = \phi \qquad x_2 = \frac{d\phi}{d\chi} \tag{12}$$

so, (11) can be written as the following system:

$$x_2 = \frac{dx_1}{d\chi} \tag{13}$$

$$\frac{dx_2}{d\chi} + 2x_2 cot\chi - 4x_1(x_1^2 - 1) = 0. \tag{14}$$

These equations can be integrated by the Runge-Kutta method [7].

We note that equation (11) is singular at $\chi = 0$ and $\chi = \pi$, so we use the following initial conditions:

$$\phi(0) = \phi_a^0 \qquad \frac{d\phi(0)}{d\chi} = \dot{\phi}_a^0 = 0 \tag{15}$$

$$\phi(\pi) = \phi_b^0 \qquad \frac{d\phi(\pi)}{d\chi} = \dot{\phi}_b^0 = 0 \tag{16}$$

So we have a differential equation (11) with boundary conditions at the two end points. It can be solved using a "shooting to a middle point" method, with ϕ_a^0 and ϕ_b^0 as shooting parameters. The main idea of this method is the following:

We perform an integration from $\chi = 0$ to $\chi = \frac{\pi}{2}$ obtaining $\phi_a(\frac{\pi}{2})$. Also we integrate from $\chi = \pi$ to $\chi = \frac{\pi}{2}$ obtaining $\phi_b(\frac{\pi}{2})$. Now, let construct the following function:

$$F(\phi_a, \phi_b) = \left[\phi_a\left(\frac{\pi}{2}\right) - \phi_b\left(\frac{\pi}{2}\right), \dot{\phi}_a\left(\frac{\pi}{2}\right) - \dot{\phi}_b\left(\frac{\pi}{2}\right)\right] \tag{17}$$

The first integration allows us to evaluate F obtaining in general $F(\phi_a^0, \phi_b^0) \neq 0$ (figure 1). We are interested in the zeros of this function, because at those points the solutions of the integration match ϕ and $\dot{\phi}$ in a smooth way. We used the Newton-Raphson method to find the zeros of a function [7], and the result is showed in figure 2.

3 Conclusion

We have shown that the theorem that precludes the existence of solitonic solutions to systems based in saclar field in $3 + 1$ or more flat space-time dimensions would be false if extended to the curve space-time case. We have done so by explicitly constructing one such solitonic solution of a simple model based on a single scalar field in the static Einstein Universe.

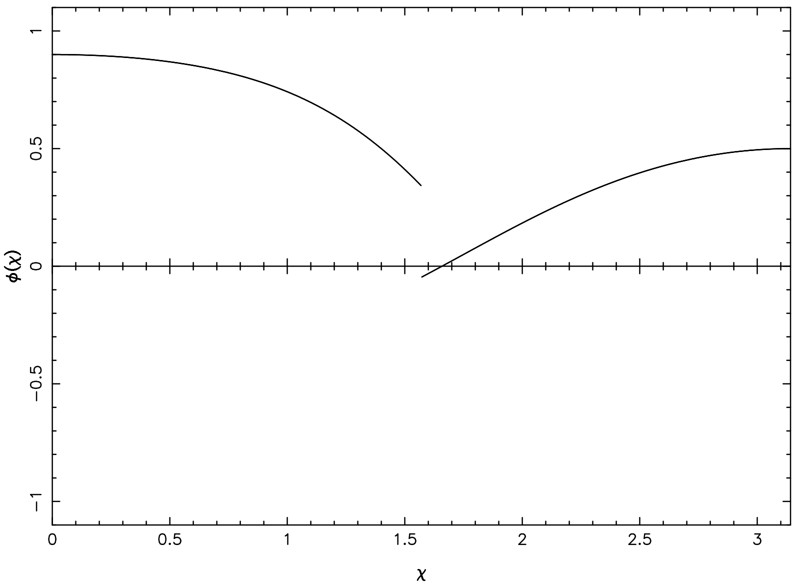

Figure 1: Integration for the initial values ϕ_a^0 and ϕ_b^0. Obviously $\phi_a(\frac{\pi}{2}) \neq \phi_b(\frac{\pi}{2})$ and $\dot{\phi}_a(\frac{\pi}{2}) \neq \dot{\phi}_b(\frac{\pi}{2})$.

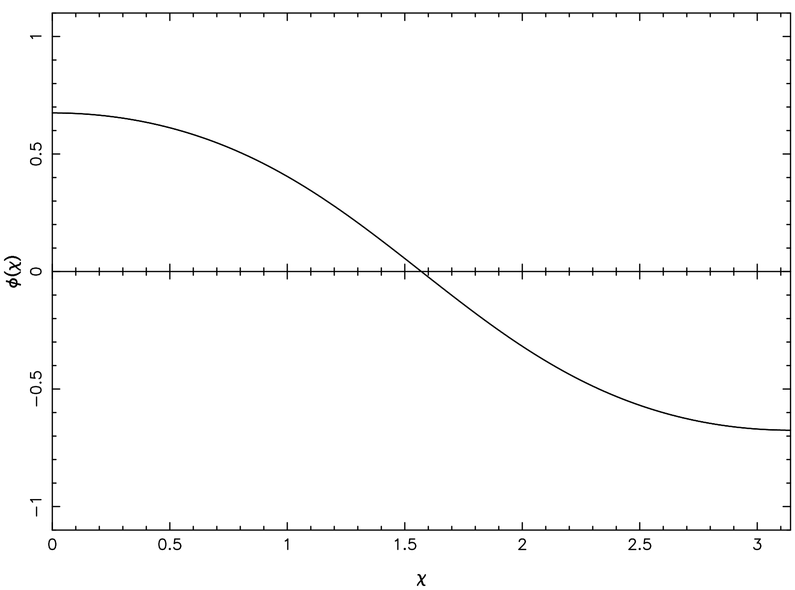

Figure 2: Integration after the application of the Newton-Raphson method. We can see that $\phi_a(\frac{\pi}{2}) = \phi_b(\frac{\pi}{2})$ and $\dot{\phi}_a(\frac{\pi}{2}) = \dot{\phi}_b(\frac{\pi}{2})$. This is a solution of the equation of motion (11).

This is another indication[3] that the interplay of curved space-time physics and soliton physics allow us for richer phenomena than each either of the two fields on its own.

4 Aknowledgments

This work was partially supported by DGAPA-UNAM grant #IN121298 and CONACyT grant #32272E.

References

[1] A.C. Scott, F.Y.F. Chiu and D.W. Mclaughlin, *Proc. I.E.E.E.* **61** (1973) 1443.

[2] A. Barone, F. Esposito, C.J. Magee and A.C. Scott, *Riv. Nuovo Cim* **1** (1971) 227.

[3] Rajaraman, R. *An introduction to Solitons and Instantons in Quantum Field Theory.* North-Holland Publishing Company, Netherlands (1982).

[4] A.A. Belavin and A.M. Polyakov, *JETP Lett* **22** (1975) 245.

[5] T.H.R. Skyrme, *Proc. R. Soc.* **A262** (1961) 233.

[6] L.D. Faddeev, *Leningrad Preprint MPI-PAE/16* (1974).

[7] Burden and Faires, *Numerical Analysis.* PWS Publishing Company, Boston, USA Fifth edition (1993).

[8] R Bartnik and J. McKinnon, *Phys.Rev.Lett.* **61** (1988) 141.

[3]The most widely knows example of this phenomenon arose in the consideration of Einstein-Yang-Mills theory where solitonic solutions have been found while it know that these are no such solutions in Yang-Mills theory in Minkowskian space-time [8].